

## A counterexample to the $a$ -‘theorem’

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**ABSTRACT:** We exhibit a renormalization group flow for a four-dimensional gauge theory along which the conformal central charge  $a$  increases. The flow connects the maximally superconformal point of an  $\mathcal{N} = 2$  gauge theory with gauge group  $SU(N+1)$  and  $N_f = 2N$  flavors in the ultraviolet, to a strongly-coupled superconformal point of the  $SU(N)$  gauge theory with  $N_f = 2N$  massless flavors in the infrared. Our example does not contradict the proof of the  $a$ -theorem via  $a$ -maximization, due to the presence of accidental symmetries in the infrared limit. Nor does it contradict the holographic  $a$ -theorem, because these gauge theories do not possess weakly-curved holographic duals.

**KEYWORDS:** Supersymmetric gauge theory, Supersymmetry and Duality, Extended Supersymmetry.

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## 1. Introduction

Zamolodchikov’s  $c$ -theorem [1] is one of the central results of two-dimensional quantum field theory. It extends the conformal central charge  $c$ , defined for conformal field theories, to a function on the space of two-dimensional field theories. This function decreases along renormalization group (RG) flows and is stationary at RG fixed points.

Over the past two decades, much effort has gone into seeking an analogue of the  $c$ -theorem in four dimensions. This effort has been complicated by the fact that comparatively little is known about nontrivial 4D conformal field theories. What is known is for the most part limited to superconformal field theories.

In two dimensions, the conformal central charge is proportional to the trace anomaly in a curved background

$$\langle T^\mu_\mu \rangle = -\frac{c}{12}R. \quad (1.1)$$

Similarly, in 4D superconformal field theories the trace anomaly depends on two constants,  $a$  and  $c$ :

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2}(\text{Weyl})^2 - \frac{a}{16\pi^2}(\text{Euler}) \quad (1.2)$$

where

$$(\text{Weyl})^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \quad (1.3)$$

$$(\text{Euler}) = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \quad (1.4)$$

Like their two-dimensional cousin, the conformal central charges  $a$  and  $c$  also appear in the stress-tensor OPE.

It is natural to ask whether  $a$ ,  $c$ , or some linear combination of them, decreases along all RG flows. Since  $a$  and  $c$  are defined by the above equation only at a CFT, we should

ask more specifically whether  $a_{\text{UV}} > a_{\text{IR}}$  or  $c_{\text{UV}} > c_{\text{IR}}$ , where for example  $a_{\text{UV}}$  denotes the value of  $a$  at the UV fixed point of the flow. (A stronger conjecture along the lines of the original  $c$ -theorem, which posits an interpolating monotonic function, will not be needed here since we shall find even this weaker conjecture to be false.) By computing  $a$  and  $c$  for various pairs of  $\mathcal{N} = 1$  superconformal field theories (SCFTs) connected by RG flows, Anselmi et al. showed that no such statement is true for  $c$ , nor for any linear combination of  $a$  and  $c$  other than, possibly,  $a$  itself [2]. In all cases they were able to check, they found that  $a_{\text{UV}} > a_{\text{IR}}$ .

Further evidence and a general argument in support of an  $a$ -‘theorem’ were given by Intriligator and Wecht [3], using the  $a$ -maximization prescription. As they noted, their argument relies on at least two assumptions: first, that no “accidental”  $U(1)$  symmetries that could potentially mix with  $U(1)_R$  appear in the IR, and second, that the local maximum of  $a$  in the UV implied by  $a$ -maximization is actually a global maximum along the flow. In the works [4–6], the effects of accidental  $U(1)$  symmetries which appear when chiral composite operators hit the unitarity bound was taken into account, but there are many known examples with  $\mathcal{N} = 2$  supersymmetry where other types of accidental  $U(1)$  symmetries appear in the IR limit. This left a big loophole in the  $a$ -theorem.

In this paper, we will give examples of RG flows that violate the conjectured  $a$ -theorem without contradicting existing results. We will employ the method of [7] for calculating  $a$  and  $c$  in  $\mathcal{N} = 2$  superconformal field theories, known as Argyres-Douglas (AD) points, which are realized as fixed points of  $\mathcal{N} = 2$  gauge theories [8]. In particular, we will study  $\mathcal{N} = 2$  SCFTs of maximal rank, which arise in  $SU(N_c)$  gauge theories with  $N_f$  fundamental flavors [9, 10]. Pairs of these SCFTs are linked by renormalization group flows, along which  $a$  should decrease if the  $a$ -theorem is valid. Instead, we will find examples of flows for which  $a_{\text{UV}} < a_{\text{IR}}$ . Specifically, these flows connect SCFTs of maximal rank, along which  $N_c$  decreases but  $N_f$  is unchanged.

The example we will consider in detail is an RG flow from the maximal-rank AD point of  $SU(N+1)$  gauge theory with  $N_f = 2N$  quark flavors to the superconformal  $SU(N)$  theory with  $2N$  massless flavors at infinitely strong coupling. We will show that the central charges  $a$  at the UV and IR endpoints of this flow are

$$a_{\text{UV}} = \frac{14N^2 + 19N}{72}, \quad a_{\text{IR}} = \frac{7N^2 - 5}{24}, \quad (1.5)$$

which violates the  $a$ -theorem when  $N \geq 4$ . In fact, in the large  $N$  limit,  $a_{\text{UV}} \sim (2/3)a_{\text{IR}}$ , giving a violation of the  $a$ -theorem of order  $N^2$ .

We will exhibit a specific deformation of the former theory which flows in the IR to the latter. The  $SU(N)$  theory in the IR possesses a marginal coupling  $\tau$ ; when the deformation of the  $SU(N+1)$  theory is small, the IR endpoint of the flow is at very strong coupling  $\tau \sim 1$ .

Another notable property of this flow is the behavior of the dimensions of the Coulomb branch operators. In the UV are they given by

$$D_{\text{UV}}(u_j) = \frac{2}{3}j, \quad (1.6)$$

while the corresponding dimensions in the IR SCFT are

$$D_{\text{IR}}(u_j) = j - 1. \quad (1.7)$$

This means that in general the dimension of each operator in the IR is significantly larger than the dimension of the corresponding operator in the UV, which is contrary to the behavior of perturbative Banks-Zaks type theories [11], where the gauge interaction plays a dominant role in creating the superconformal point. It is reasonable to attribute at least some part of the increase in  $a$  to this large increase in anomalous dimensions. An indication that this is indeed the case comes from the general formula [12, 7]

$$4(2a - c) = \sum_{i=1}^r (2D(u_i) - 1) \quad (1.8)$$

valid in all 4d  $\mathcal{N} = 2$  SCFTs under consideration. Inserting the above anomalous dimensions into the sum on the right-hand side of this equation, it is easy to see that the combination  $2a - c$ , like  $a$ , increases by a factor of approximately  $3/2$  along the flow in question.

There are many other examples of flows of this type. From the formula for  $a$  that we will derive in section 3, it follows directly that for any  $N_c$  and  $N_f$  satisfying

$$2N_c > N_f > \frac{1}{2}(\sqrt{21} - 3)N_c \approx 0.79N_c, \quad (1.9)$$

with  $N_c$  sufficiently large, there is a flow between Argyres-Douglas points of the  $\text{SU}(N_c+1)$  and  $\text{SU}(N_c)$  gauge theories, both with  $N_f$  flavors, along which  $a$  increases.

The first question raised by our result is, how do our examples avoid falling into one of the classes of theories for which the  $a$ -theorem has been established. First, as has already been mentioned, the proof using the method of  $a$ -maximization [4–6] relies on a strong assumption about the existence of accidental symmetries. In our case, this assumption is clearly violated: the  $\text{U}(1)_R$  symmetry is broken throughout the RG flow, and the  $\text{U}(1)_R$  that appears in the IR has nothing to do with the UV  $R$ -symmetry; it is totally accidental. Second, our examples do not contradict the holographic derivation of the  $a$ -theorem because the CFTs involved have no holographic duals. Indeed, any CFT with a weakly curved gravity dual has  $a$  and  $c$  both of order  $N^2$  and  $a - c$  of order at most  $N$  [13], whereas in our case  $a - c$  is of order  $N^2$ . Third, it might seem rather surprising to the reader that counterexamples can be found within a class of SCFTs that was first discussed in 1996 [10], well before many of the more modern analyses of the  $a$ -theorem [3, 14]. The reason these counterexamples were not recognized sooner is simply that there was no method to calculate  $a$  for these theories before the work [7].

We begin by reviewing in section 2 the method developed in [7] for obtaining the central charges  $a$  and  $c$  of  $\mathcal{N} = 2$  SCFTs. We apply this method in section 3 to the maximal-rank superconformal points of  $\text{SU}(N_c)$  gauge theory with  $N_f$  flavors, which were first studied in [10]. In section 4 we will find that there is a violation of the  $a$ -theorem within this class of SCFTs, and we will take some care to establish that there is indeed an RG flow between the two particular SCFTs involved in our main example. We close the paper in section 5 with a discussion of our results. We briefly summarize the history of the four-dimensional analogue of the  $c$ -theorem in appendix A.

## 2. $a$ and $c$ for $\mathcal{N} = 2$ gauge theories

Recently we developed a method for calculating central charges of  $\mathcal{N} = 2$  gauge theories, by relating them to  $R$ -symmetry anomalies in the corresponding topological field theory [7]. Here we will review the method, in order to set the stage for its application in the next section to the calculation of  $a$  and  $c$  for  $SU(N)$  gauge theories with hypermultiplet matter.

We begin by recalling the relation of  $a$  and  $c$  to the anomalous conservation law of the  $U(1)_R$  current in  $R^\mu$  any  $\mathcal{N} = 2$  field theory [15], in the presence of a background metric and a background  $SU(2)_R$  gauge field  $F_{\mu\nu}^a$ :

$$\partial_\mu R^\mu = \frac{c-a}{8\pi^2} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \frac{2a-c}{8\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \quad (2.1)$$

where

$$\tilde{F}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}^a, \quad \tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma}. \quad (2.2)$$

According to the well-known construction of topological gauge theories, in backgrounds where the  $SU(2)_R$  gauge field is equal to the self-dual part of the curvature, i.e.

$$F_{\mu\nu}^a t_{\rho\sigma}^a = \frac{1}{2} (R_{\mu\nu\rho\sigma} + \tilde{R}_{\mu\nu\rho\sigma}) \quad (2.3)$$

with the 't Hooft symbol  $t_{\rho\sigma}^a$ , correlation functions of physical operators depend only on the topology of the background manifold. Substituting this condition into (2.1) gives the anomaly equation in the topological background, which when integrated over the 4-manifold gives the total  $R$ -charge of the vacuum

$$\Delta R = 2(2a - c)\chi + 3c\sigma \quad (2.4)$$

in terms of the Euler characteristic  $\chi$  and the signature  $\sigma$  of the manifold.

Thus to determine  $a$  and  $c$ , it suffices to be able to compute the dependence of  $\Delta R$  on the topology of the background. This information is encoded in the path integral measure, which for a topological gauge theory takes the form

$$[d\mu] A^\chi B^\sigma \quad (2.5)$$

The factor  $[d\mu]$  is the measure for the  $r$  vector multiplets, which at a generic point in moduli space are the only massless modes. The measure factors  $A$  and  $B$  depend holomorphically on the Coulomb branch moduli and are associated with the additional massless states that appear on special loci of complex codimension 1 and higher.

The  $R$ -charge of the vacuum can be directly read off from the measure (2.5)

$$\Delta R = \chi R(A) + \sigma R(B) + \frac{\chi + \sigma}{2} r. \quad (2.6)$$

Here  $R(A)$  and  $R(B)$  denote the  $R$  charges of  $A$  and  $B$ , and the last term is the contribution of the generically massless vector multiplets. Comparing with (2.4), we find the following general expressions for  $a$  and  $c$ :

$$a = \frac{1}{4} R(A) + \frac{1}{6} R(B) + \frac{5}{24} r, \quad c = \frac{1}{3} R(B) + \frac{1}{6} r \quad (2.7)$$

Finding  $a$  and  $c$  is thus reduced to calculating the  $R$ -charge of the functions  $A$  and  $B$ . In general, these functions are believed to take the form [16–20]

$$A(u) = \alpha \left[ \det \frac{\partial u_i}{\partial a^I} \right]^{1/2}, \quad B(u) = \beta \Delta^{1/8}. \quad (2.8)$$

Here,  $u_i$  are gauge- and monodromy- invariant coordinates on the Coulomb branch,  $a^I$  are special coordinates, and  $\Delta$  is the physical discriminant of the Seiberg-Witten curve.  $\alpha$  and  $\beta$  are prefactors independent of the  $u_i$  which can in principle depend on the mass parameters. The functions in (2.8) are readily computable in the vicinity of many superconformal points of  $\mathcal{N} = 2$  gauge theories.

The  $R$ -charges of  $A$  and  $B$  can be written in terms of the  $R$ -charges  $R(u_i)$ , or their dimensions  $D(u_i)$ , which satisfy

$$R(u_i) = 2D(u_i) \quad (2.9)$$

by virtue of superconformal symmetry. Finally, the dimensions  $D(u_i)$  can be obtained from the scaling form of the Seiberg-Witten curve by demanding that the dimension of the Seiberg-Witten differential  $\lambda_{\text{SW}}$  be one. This completes the calculation of the central charges  $a$  and  $c$ .

### 3. Superconformal points of $\mathcal{N} = 2$ $\text{SU}(N_c)$ with quarks

We will now apply the method described in the previous section to calculate the conformal central charges of an infinite family of 4D SCFTs, which correspond to Argyres-Douglas points [8] in the Coulomb branch of  $\mathcal{N} = 2$   $\text{SU}(N_c)$  gauge theory with  $N_f$  flavors of fundamental quarks [10]. At these points, a maximal set of mutually nonlocal dyons becomes massless, and the Seiberg-Witten curve develops a singularity of maximal rank. Here the rank of a superconformal theory signifies the minimal number of  $\text{U}(1)$  vector multiplets to which the set of dyons with degenerating mass couple electrically or magnetically. We will restrict our discussion to theories with even  $N_f \equiv 2n_f$ .

To locate these points, we start with the Seiberg-Witten curve for  $\mathcal{N} = 2$  supersymmetric  $\text{SU}(N_c)$  gauge theory with  $N_f < 2N_c$  fundamental hypermultiplets of equal mass, which has the form [21, 22]

$$y^2 = P(x)^2 - \Lambda^{2N_c - N_f} (x + m)^{N_f} \quad (3.1)$$

where

$$P(x) = x^{N_c} + u_2 x^{N_c - 2} + u_3 x^{N_c - 3} \dots + u_{N_c}. \quad (3.2)$$

$\Lambda$  is the dynamically generated scale of the gauge theory, and one can identify  $u_j$  with the composite operator  $\text{tr } \phi^j$  in the semiclassical regime. The Seiberg-Witten differential is

$$\lambda_{\text{SW}} = x d \log \frac{1 - y/P}{1 + y/P}. \quad (3.3)$$

To reach the superconformal point of maximal rank, we first choose the moduli  $u_i$  so that

$$P(x) = (x + m)^{n_f} C_{N_c - n_f}(x) \quad (3.4)$$

where  $C_{N_c-n_f}(x)$  is a polynomial of degree  $N_c - n_f$ . Then the curve becomes

$$y^2 = (x+m)^{N_f}(C_{N_c-n_f}(x) - \Lambda^{N_c-n_f})(C_{N_c-n_f}(x) + \Lambda^{N_c-n_f}) \quad (3.5)$$

The roots of  $C_{N_c-n_f}$  can further be adjusted by tuning the remaining moduli and  $m$ ; we can use this freedom to set  $C_{N_c-n_f}(x) = (x+m)^{N_c-n_f} - \Lambda^{N_c-n_f}$ , giving a singularity of maximal degree

$$y^2 = (x+m)^{N_c+n_f}((x+m)^{N_c-n_f} - 2\Lambda^{N_c-n_f}). \quad (3.6)$$

For  $N_c - n_f \geq 2$  this procedure leads to the choice

$$m = 0, \quad P(x) = x^{N_c} - \Lambda^{N_c-n_f} x^{n_f}. \quad (3.7)$$

It is known that another branch of the moduli space touches the Coulomb branch at this point. In the terminology of [23], this is a special point on the non-baryonic Higgs branch root with extra massless monopoles, whose generic massless spectrum is that of a  $U(n_f)$  gauge theory with  $N_f = 2n_f$  quarks. The Higgs branch emanating from this point has quaternionic dimension  $n_f^2$ .

For  $N_c - n_f = 1$  we need to choose

$$m = \frac{\Lambda}{N_c}, \quad P(x) = (x+m)^{N_c} - \Lambda x^{N_c-1} \quad (3.8)$$

in order to guarantee that the coefficient of the  $x^{N_c-1}$  term of  $P(x)$  vanishes. Again this point lies at the root of a non-baryonic Higgs branch of quaternionic dimension  $n_f^2$ .

The SCFT at this point was first studied by [10] and denoted by the symbol  $M_{N_c+n_f}^{N_f}$ . By expanding the one-form (3.3) around this point and demanding  $D(\lambda_{\text{SW}}) = 1$ , the authors of [10] found the scaling dimensions

$$D(x) = \frac{2}{N_c - n_f + 2}, \quad D(u_j) = jD(x). \quad (3.9)$$

When  $N_c + n_f$  is odd, there are  $r = (N_c + n_f - 1)/2$  pairs of special coordinates  $a^I$  which become zero at the superconformal point, i.e. the rank of this theory is  $r$ . The Coulomb branch operators with dimension  $> 1$  are  $u_j$  with  $r - n_f + 2 \leq j \leq N_c$ . So there are  $N_c - (r - n_f + 2) + 1 = r$  of them, as expected. As was argued in [10], there are loci in the  $SU(N_c)$  gauge theory with  $N_f$  quarks where the low-energy theory becomes superconformal with non-maximal rank  $r' < r$ . These non-maximal superconformal points with rank  $r'$  are known to be equivalent to the maximal-rank superconformal points of the  $SU(N_c - 2r + 2r')$  gauge theory with  $N_f$  quarks. Thus the maximal superconformal points are naturally labeled by their rank  $r$  and the number of flavors  $N_f$ , and we will express the central charges in terms of these quantities.

The dimensions (3.9) determine the  $R$ -charge of the measure factor  $A$ :

$$R(A) = \sum_{j=r-n_f+2}^{N_c} (D(u_j) - 1) = \frac{r^2}{2r - 2n_f + 3}. \quad (3.10)$$

Also, the discriminant of the curve is

$$\Delta = B^8 = \prod_{i>j} (e_i - e_j)^2, \quad (3.11)$$

where  $e_i$  ( $i = 1, \dots, 2N$ ) are the branch points of the curve (3.1). Note that only  $2r + 1$  of the branch points  $e_i$  are small and have the same dimension as  $\delta x$ . Thus we have

$$R(B) = \frac{r(2r+1)}{2r-2n_f+3}. \quad (3.12)$$

Therefore the central charges are given by the formula (2.7) :

$$a = \frac{r(24r-10n_f+19)}{24(2r-2n_f+3)}, \quad c = \frac{r(6r-2n_f+5)}{6(2r-2n_f+3)}. \quad (3.13)$$

Similarly, when  $N_c + n_f$  is even, there are  $r = (N_c + n_f)/2 - 1$  pairs of special coordinates  $a^I$  which become zero. The Coulomb branch operators with dimension  $> 1$  are  $u_j$  with  $r - n_f + 3 \leq j \leq N_c$ . So there are again  $N_c - (r - n_f + 2) + 1 = r$  of them, as expected. We have

$$R(A) = \frac{r(r+1)}{2(r-n_f+2)}, \quad R(B) = \frac{(r+1)(2r+1)}{2(r-n_f+2)}. \quad (3.14)$$

Thus the central charges are

$$a = \frac{12r^2 + (19 - 5n_f)r + 2}{24(r - n_f + 2)}, \quad c = \frac{3r^2 + (5 - n_f)r + 1}{6(r - n_f + 2)}. \quad (3.15)$$

Note that in all cases, the ratio of the central charges satisfies the inequality

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{5}{4} \quad (3.16)$$

which was discussed in [24, 7].

Let us study the case  $N_f = 2N_c$  separately. The curve is given by [21]

$$y^2 = P(x)^2 - f(\tau)Q(x), \quad (3.17)$$

where

$$P(x) = x^{N_c} + u_2 x^{N_c-2} + u_3 x^{N_c-3} \dots + u_{N_c}, \quad (3.18)$$

$$Q(x) = \prod_{a=1}^{2N_c} (x - m - 2g(\tau)m). \quad (3.19)$$

Here,  $f(\tau)$  and  $g(\tau)$  are certain modular functions of the complexified gauge coupling  $\tau$ . The superconformal point of maximal rank  $r = N_c - 1$  is reached by scaling the mass  $m$  and vevs  $u_k$  to zero with canonical scaling dimensions. This conformal point has one marginal coupling  $\tau$ . The central charges we obtain from the formula (2.7) are

$$a = \frac{7}{24}N_c^2 - \frac{5}{24}, \quad c = \frac{1}{3}N_c^2 - \frac{1}{6}. \quad (3.20)$$

Alternatively, the central charges can be computed in the weak-coupling limit  $\text{Im } \tau \rightarrow \infty$  as sums of contributions of free fields. Since  $a$  and  $c$  must be independent of  $\tau$ , the free-field answer should, and does, agree with (3.20).



#### 4. Violation of the $a$ -‘theorem’

Let us now compare the central charges  $a$  of two of the SCFTs studied above, the  $SU(N+1)$  theory and the  $SU(N)$  theory, both with  $N_f = 2N$  quark flavors. We claim that there is a renormalization group flow from the former to the latter. The values of the central charge  $a$  at the UV and IR endpoints of this flow are then

$$a_{\text{UV}} = \frac{14N^2 + 19N}{72}, \quad a_{\text{IR}} = \frac{7N^2 - 5}{24}, \quad (4.1)$$

which violates the  $a$ -theorem when  $N \geq 4$ . In fact, in the large  $N$  limit  $a_{\text{UV}} \sim (7/36)N^2$  and  $a_{\text{IR}} \sim (7/24)N^2$  so we have  $a_{\text{UV}} \sim (2/3)a_{\text{IR}}$ , which amounts to an  $a$ -theorem violation of  $\mathcal{O}(N^2)$ .

In order to confirm that we have indeed found a counterexample to the  $a$ -theorem, we need to establish that there is in fact an RG flow starting from the maximal superconformal point of the  $SU(N+1)$  theory with  $2N$  quarks in the ultraviolet, to the  $SU(N)$  theory with the same number of quarks.

To this end, let us study the  $SU(N+1)$  theory with  $2N$  quarks in more detail. When  $P(x) = (x+m)^N(x-Nm)$ , the curve becomes

$$y^2 = (x+m)^{2N}(x-Nm+\Lambda)(x-Nm-\Lambda). \quad (4.2)$$

Therefore the maximal-rank superconformal point occurs when  $-Nm + \Lambda = m$ , as also discussed in the previous section. We parameterize the deviation from this value of  $m$  as

$$-Nm + \Lambda = m + \delta m \quad (4.3)$$

and expand  $P(x)$  around the superconformal point as

$$P(x) = (x+m)^N(x-Nm) + \tilde{u}_2(x+m)^{N-1} + \dots \quad (4.4)$$

$$= \tilde{x}^N(\tilde{x} + \delta m - \Lambda) + \tilde{u}_2\tilde{x}^{N-1} + \tilde{u}_3\tilde{x}^{N-2} + \dots \quad (4.5)$$

where  $\tilde{x} = x + m$ . Now the curve takes the form

$$y^2 = [x^N(x + \delta m) + u_2x^{N-1} + u_3x^{N-2} + \dots] \times [x^N(x + \delta m - 2\Lambda) + u_2x^{N-1} + u_3x^{N-2} + \dots] \quad (4.6)$$

where we have dropped all tildes. Note that we have not yet made any approximations.

Let us now study the behavior of the curve (4.6) very close to the superconformal point, in the limit

$$|x| \sim |u_j|^{1/j} \sim |\delta m| \ll |\Lambda| \quad (4.7)$$

Then the curve is approximately

$$y^2 \sim -2\Lambda x^N [x^N(x + \delta m) + u_2x^{N-1} + u_3x^{N-2} + \dots]. \quad (4.8)$$

This describes a general deformation of the maximal superconformal point.

If we instead take the scaling limit

$$|x| \sim |u_j|^{1/(j-1)} \ll |\delta m| \ll |\Lambda|, \quad (4.9)$$

then the curve becomes approximately

$$y^2 \sim [x^N(\delta m) + u_2 x^{N-1} + u_3 x^{N-2} + \dots] \times [x^N(\delta m - 2\Lambda) + u_2 x^{N-1} + u_3 x^{N-2} + \dots] \quad (4.10)$$

$$= [(\delta m - \Lambda)x^N + u_2 x^{N-1} + u_3 x^{N-2} + \dots]^2 - \Lambda^2 x^{2N} \quad (4.11)$$

Absorbing the factor  $(\delta m - \Lambda)$  into  $y$ , shifting  $x$  to eliminate the second term in brackets, and redefining  $\hat{u}_j \equiv u_{j+1}/(\delta m - \Lambda)$ , we finally obtain

$$y^2 \sim [x^N + \hat{u}_2 x^{N-2} + \dots]^2 - \left( \frac{\Lambda}{\delta m - \Lambda} \right)^2 (x - \mu)^{2N} \quad (4.12)$$

which we recognize as the curve (3.17) of the  $SU(N)$  theory with  $2N$  flavors of mass

$$\mu \equiv \frac{u_2}{N(\delta m - \Lambda)} \quad (4.13)$$

at a particular value of the coupling  $\tau$  depending on the ratio of  $(\delta m - \Lambda)$  and  $\Lambda$ . The mass is automatically zero when  $u_2 = 0$ , and then the change in  $\delta m$  directly translates to a change in  $\tau$ . Using the explicit form of the modular function  $f(\tau)$  [21], this value is found to be close to the infinite coupling point  $\tau = 1$ .

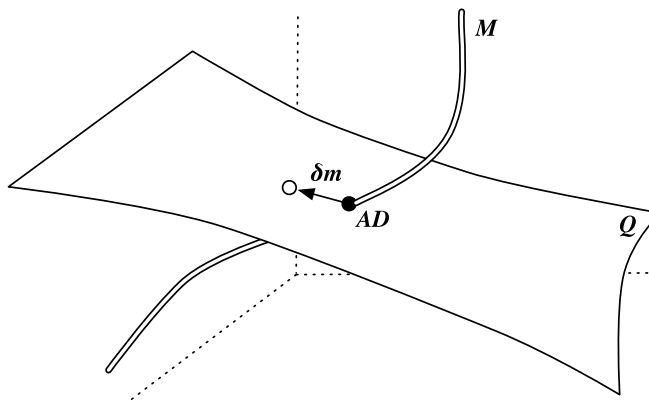
The subspace of the  $SU(N)$  moduli-parameter space generated by deforming the  $\tau = 1$  superconformal point by  $\delta m$  extends out to the semiclassical region. There, it can be matched onto the moduli space of the parent  $SU(N+1)$  theory with  $2N$  quarks of mass  $m$ , Higgsed down to the  $SU(N)$  theory with  $2N$  flavors by the adjoint scalar vev

$$\langle \phi \rangle = \text{diag} (m, m, \dots, m, -Nm). \quad (4.14)$$

A schematic picture of the moduli-parameter space of the  $SU(N+1)$  gauge theory is depicted in figure 1. There, the subspace  $Q$  is the locus generated by  $\delta m$  inside the space of  $m, u_2, \dots, u_{N+1}$  where the low energy theory contains massless  $SU(N)$  gauge bosons and  $2N$  massless flavors. (It is slightly unconventional to depict the mass parameter  $m$  and the moduli  $u_j$  together in the same moduli-parameter space, but as was explained in [9], this is a natural point of view to take in our situation.) Indeed, one may think of the space of  $m$  and  $u_j$  as the moduli space of a  $U(N+1)$  gauge theory with  $2N$  flavors. The locus  $M$ , where an extra magnetically charged state becomes massless, intersects with  $Q$  at the AD point. This depiction is highly schematic, e.g. in that the intersection is not transversal as in the figure and is far more complicated in reality, as was well-illustrated for the case  $N = 2$  in the original paper [10]. (See the discussion following eq. (33) therein.)

With these preparations, fix  $|\delta m| \ll |\Lambda|$ , and consider the following three regimes of vevs of  $u_j$ 's:

$$\mathbf{1.} \quad |u_j| \sim |\Lambda|^j; \quad \mathbf{2.} \quad |u_j| \sim |\delta m|^j; \quad \mathbf{3.} \quad |u_j| \sim \epsilon^{j-1} |\delta m|, \quad \epsilon \ll |\delta m|$$



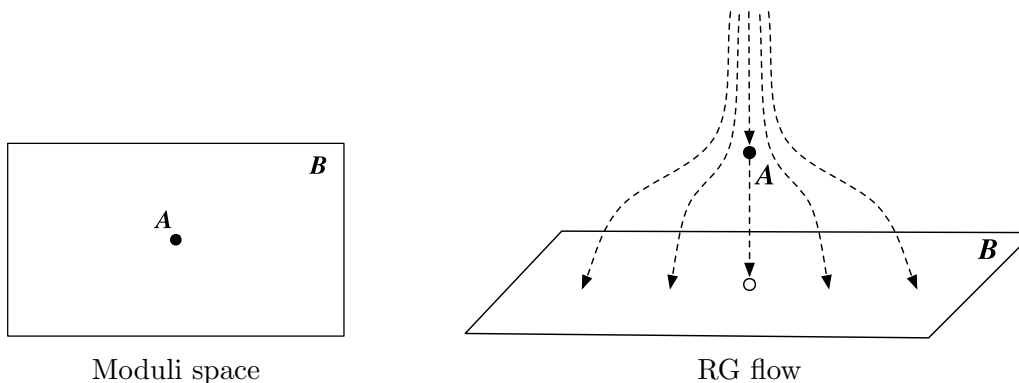
**Figure 1:** Depiction of the moduli-parameter space of the  $SU(N+1)$  theory with  $2N$  flavors.  $Q$  is the subspace where the low energy theory has unbroken  $SU(N)$  gauge bosons and  $2N$  massless flavors. The locus  $M$  where an extra monopole becomes massless intersects with  $Q$  at the  $AD$  point.

For  $u_j$  in regime 1, we are at a generic point in the moduli space of the  $SU(N+1)$  theory with  $2N$  flavors with no particular interest. The special coordinates  $a^I$ , or equivalently the masses of the BPS solitons, are all of order  $\Lambda$ . As we lower the  $u_j$  and enter regime 2, the system exhibits the scaling of the maximal  $AD$  point of  $SU(N+1)$  theory with  $2N$  flavors, i.e. the dimension of  $u_j$  is  $2j/3$ . The low-lying spectrum of BPS masses is that of the maximal  $AD$  point, of order  $(\delta m)^{3/2}$ . When we further lower  $u_j$  to regime 3, the scaling dimensions are those of the  $SU(N)$  theory with  $2N$  massless flavors, and  $u_j$  has canonical dimension  $j - 1$ . The low-lying BPS solitons have masses of order  $\epsilon$ .

Our discussion up to this point has just been a standard analysis of a trajectory through the moduli space of an  $\mathcal{N} = 2$  gauge theory. In other words, we have studied how the couplings and BPS masses behave under a particular change of the vevs. While this trajectory is reminiscent of an RG flow, in that the energy scale defined by the vevs is changing, the true RG flow is along an extra direction which is not tangent to the moduli space. Indeed, the solution of  $\mathcal{N} = 2$  gauge theories via Seiberg-Witten curves describes the moduli-dependence of infrared fixed points, which by definition do not flow.

Let us first understand how RG flow is manifested in the simpler case of the pure  $SU(2)$  gauge theory. The moduli space is parameterized by a single vev  $u$ , and the low energy theory at generic values of  $u$  is a free  $U(1)$  gauge theory with coupling  $\tau(u)$ . The moduli space thus consists of a family of trivial conformal fixed points parameterized by the marginal coupling  $\tau$ . For special values of  $u$ , say  $u = \Lambda^2$ , the low-energy fixed point theory includes an extra massless hypermultiplet and the gauge coupling vanishes. This fixed point can be deformed by the operator  $u$  which makes the hypermultiplet massive; the endpoint of the flow from the deformed theory is a free  $U(1)$  gauge theory.

This situation is heuristically depicted in figure 2. On the left-hand side, the moduli space is split into a special point  $A$ , where the spectrum includes an extra massless hypermultiplet, and the generic region  $B$ , where the low-energy limit contains only a  $U(1)$



**Figure 2:** Distinction between the moduli space and the RG flow. See the text for explanation.

gauge field. The right-hand side is a cartoon of the RG flow: the point  $A$  and the family  $B$  are low-energy endpoints of flows, and are by definition conformal. They are embedded in a larger space of non-conformal theories, through which the RG transformation generates flows. The special point  $A$  can be viewed either as an IR fixed point of a flow from the microscopic theory, or as a UV fixed point whose relevant deformations generate flows to IR fixed point theories in  $B$ . At the conformal point  $A$ , the coupling of the U(1) gauge field is strictly zero. Therefore, the flow starting exactly at  $A$  ends at the zero coupling limit of the family  $B$ . If the RG flow starts slightly away from  $A$ , i.e. if the gauge coupling of the theory is not strictly zero, then the endpoint of the flow after the decoupling of the hypermultiplet has nonzero coupling constant which thus corresponds to a generic point of the family  $B$ .

We will argue that the RG flow between the AD point and the space  $Q$  is quite analogous to the example just discussed, with  $A$  representing the AD point and  $B$  corresponding to the moduli space  $Q$ . Let us fix  $u_j^{(0)}$  and  $\delta m^{(0)}$  to be sufficiently small, but finite, compared to the dynamical scale of the gauge theory  $|\Lambda|$ . We then consider the parameterized locus of deformations away from the AD point

$$\delta m = \lambda \delta m^{(0)}, \quad u_j = \lambda^j u_j^{(0)}. \quad (4.15)$$

We wish to study the RG flow which passes through a point on this locus. At such a point, the lightest massive BPS states set a mass scale  $M(\lambda)$ , which we can assume is much less than the dynamical scale  $|\Lambda|$  of the gauge theory. Since the scaling dimension of  $\delta m$  close to the AD point is  $2/3$ , it follows that  $M(\lambda) \sim \lambda^{3/2}$  as  $\lambda \rightarrow 0$ . Now consider an RG scale  $\Lambda_{\text{RG}}$  in the range

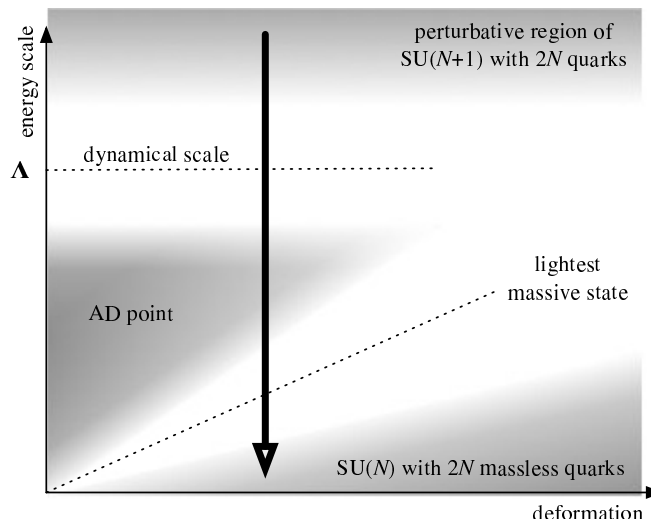
$$M(\lambda) \ll \Lambda_{\text{RG}} \ll |\Lambda|, \quad (4.16)$$

At such scales the BPS states, which become exactly massless at the AD point, are effectively massless, and the theory is effectively equivalent to the superconformal AD theory.

Let us next consider the following trajectory in the moduli space

$$\delta m = \delta m^{(0)}, \quad u_j = \epsilon^{j-1} u_j^{(0)}, \quad (4.17)$$

where we take  $\epsilon$  to be very small. Then the BPS states have two typical mass scales,  $M$  determined by  $\delta m$  and  $\mu(\epsilon)$  determined by  $u_j$ 's. By the analysis of the Seiberg-Witten



**Figure 3:** Schematic RG flow. For scales above  $\Lambda$ , the system is in the perturbative regime of the parent  $SU(N+1)$  theory with  $2N$  flavors. Below  $\Lambda$ , the RG flow passes close to the maximal superconformal point. The deformation  $\delta m$  becomes important once the energy scale is comparable to the masses of light BPS states, and the theory flows to the  $SU(N)$  theory with  $2N$  massless flavors in the far IR. The system is nearly conformal in the three shaded regions.

curve, we know  $\mu(\epsilon)$  scales as  $\mu(\epsilon) \sim \epsilon$  when  $\epsilon \rightarrow 0$ . Therefore the system viewed at the RG scale  $\Lambda_{RG}$  in the range

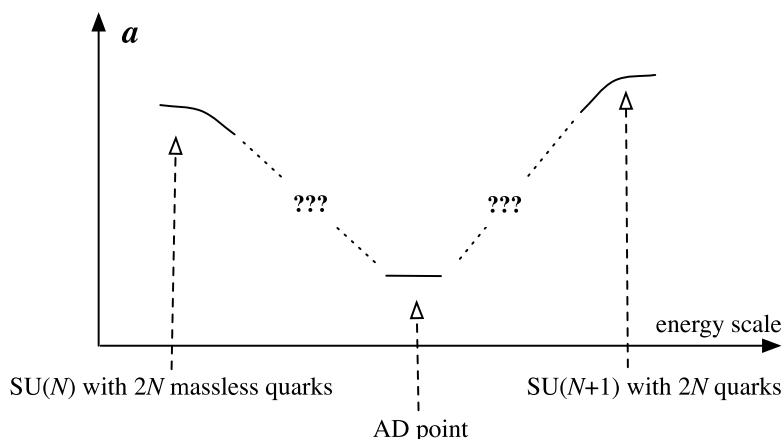
$$\mu(\epsilon) \ll \Lambda_{RG} \ll M \quad (4.18)$$

is effectively equivalent to the superconformal  $SU(N)$  theory with  $2N$  massless flavors. Below  $\mu(\epsilon)$ , the vevs  $u_j$  break the  $SU(N)$  theory down to a theory with decoupled  $U(1)$  vector multiplets. Our interpretation of the RG flow is summarized in figure 3.

To recapitulate our discussion, the  $SU(N+1)$  theory with  $2N$  quarks and with  $\delta m \neq 0$  has the following evolution along the RG flow: In the extreme UV it starts as a perturbative gauge theory with gauge group  $SU(N+1)$  and  $2N$  quarks. It becomes strongly coupled at a scale of order  $\Lambda$ , and gets attracted to the maximal AD point. It then starts to be affected by the small deformation  $\delta m \neq 0$ . This deformation is relevant, because the parameter  $\delta m$  has dimension  $2/3$  and the corresponding operator which  $\delta m$  multiplies in the Lagrangian has dimension  $4/3$ . Far below that scale, the flow eventually ends at the  $SU(N)$  theory with  $2N$  massless flavors, close to the infinite coupling point  $\tau = 1$ . Combined with the calculation of  $a$  we performed in the last section, this RG flow establishes the violation of the  $a$ -theorem. The behavior of  $a$  is shown in figure 4. This figure is again highly schematic because we do not have a proper interpolating  $a$ -function at intermediate scales.

## 5. Discussion

The careful analysis of the preceding section establishes that there exists an RG flow from the AD point of the  $SU(N+1)$  theory with  $2N$  quarks to the  $SU(N)$  theory with  $2N$  massless quarks. If one still wishes to rescue the  $a$ -theorem, one needs to scrutinize the



**Figure 4:** Schematic graph showing the change in the central charge  $a$ .

calculation of the central charges in section 3, which is an application of the authors' recent work [7].

One question that could be asked concerns the role of the Higgs branch. In our method, the SCFT point is studied by slightly moving away from it along the Coulomb branch. But we know that a non-baryonic Higgs branch with quaternionic dimension  $n_f^2$  emanates from the SCFT point, as was recalled in section 3. Is the contribution from this branch correctly accounted by this method? The answer is yes; the factors  $A(u)$  and  $B(u)$  on the Coulomb branch arise from integrating out the massive states in the theory. The fact that  $A(u)$  and  $B(u)$  have zeroes of order  $N^2$  at the SCFT point signifies that the number of light degrees of freedom is of order  $N^2$ . In [7] this method was applied to the  $\text{USp}(2N)$  theory with  $N_f = 1, 2, 3$  quarks, in addition to a hypermultiplet in the antisymmetric representation, which also has a large Higgs branch. The theory has an F-theoretic holographic dual and the Higgs branch corresponds to the absorption of D3-branes onto a stack of 7-branes as instantons. The fact that the calculation based on this method completely reproduces the central charges found using holography in [25] demonstrates its overall consistency. There is no problem regarding the possible dependence of the measure factors  $A, B$  on the Higgs branch vevs either, because we always work slightly away from the SCFT point along the Coulomb branch, where there are no massless hypermultiplets.

Another question, also related to the Higgs branch, concerns the identification of the  $\text{U}(1)_R$  symmetry. The superconformal  $\text{U}(1)_R$  charges of the vector multiplet scalars  $u_j$  are fixed by the Seiberg-Witten differential, but there could in principle be extra non-R  $\text{U}(1)$  symmetries under which the  $u_j$  are neutral, which could mix into the  $\text{U}(1)_R$ . An example of such a  $\text{U}(1)$  symmetry is the  $\text{U}(1)_B$  symmetry which acts on the Higgs branch as the  $\text{U}(1)$  part of the  $\text{U}(N_f)$  flavor rotation. However, this  $\text{U}(1)_B$  is vector-like, and thus cannot mix with the  $\text{U}(1)_R$  symmetry [3]. This conclusion can also be drawn from the relation (2.1): the central charges are encoded in the 't Hooft anomalies of the forms  $\text{U}(1)$ -gravity-gravity and  $\text{U}(1)$ - $\text{SU}(2)_R$ - $\text{SU}(2)_R$ , but  $\text{U}(1)_B$  has neither type of anomaly. Thus, it

cannot contribute to  $a$  or  $c$  even if it mixes with the  $U(1)_R$  symmetry which we identified.

As was discussed in [23], there is no other  $U(1)$  symmetry which acts on the Higgs branch. The next possibility to be ruled out is the existence of an accidental, non-R, chiral  $U(1)$  symmetry (let us call it  $T$ ) which appears at the SCFT point, under which the  $u_j$  are neutral. But we find the existence of such a symmetry highly unlikely: a small, generic deformation along the Coulomb branch from the SCFT point, which is generated by giving vevs to the  $u_j$ , does not break  $T$  because the  $u_j$  are neutral under  $T$ . In other words, this unbroken symmetry  $T$  should also be present slightly away from the SCFT point, where it can act only on massive states because the only massless states away from the SCFT point are free vector multiplets. This is a contradiction, because a chiral symmetry  $T$  can only act on massless states.<sup>1</sup>

A rather trivial question the reader might have is about the decoupled sector in the infrared. At the AD points with  $N_f < 2N_c$ , the number  $r$  of vector multiplets which couple to mutually nonlocal states is smaller than the rank  $N - 1$  of the original gauge theory, and so there are  $N - 1 - r$  decoupled free vector multiplets. In the calculation in section 3 we actually did not include the contribution of these decoupled vector multiplets to  $a$  and  $c$ . Could their inclusion change the value of  $a$  sufficiently so that the  $a$ -theorem is saved? The answer is no — including them makes the violation of the  $a$ -theorem worse, not better. Furthermore, the violation we found is of order  $O(N^2)$  and there are at most  $O(N)$  free decoupled vector multiplets, so they cannot change the big picture.

Before closing the paper we would like to briefly reiterate how the established cases of the  $a$ -theorem fail to apply to our counterexample. Proofs based on holography are not applicable here, because our AD points do not have holographic duals which are weakly curved. Indeed, we found  $a$  and  $c$  both to be of order  $N^2$ , but  $a/c \rightarrow 7/8$  in the large  $N$  limit, whereas  $a/c \rightarrow 1$  in the large  $N$  limit of any gauge theory with a weakly-curved  $AdS_5$  dual. Another class of proofs, based on a combination of  $a$ -maximization and 't Hooft anomaly matching, do not apply, because there are no other symmetries with which  $U(1)_R$

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<sup>1</sup>Let us apply this argument to a trivial SCFT point whose low energy content is a  $U(1)$  gauge theory coupled to a hypermultiplet, formed by two  $\mathcal{N} = 1$  chiral superfields  $(q, \tilde{q})$ . Denote the vector multiplet scalar by  $\phi$ . The theory is completely free in the IR, so there is an accidental chiral  $U(1)$  symmetry, call it  $T'$ , under which both  $q$  and  $\tilde{q}$  have charge  $+1$ , and  $\phi$  is neutral. This  $T'$  is broken along the Coulomb branch, which at first appears to contradict the argument presented above.

The point is that this  $U(1)$  symmetry does not commute with the  $SU(2)_R$  symmetry. Existence of such symmetries is usually forbidden by the Haag-Lopuszański-Sohnius theorem, which is not applicable for a free theory. Strictly speaking, there is no definite proof of this theorem or of the Coleman-Mandula theorem for an interacting CFT, as is mentioned in the footnote on p. 13 of Weinberg's textbook [26]. This is because the proofs of these theorems are phrased in terms of the S-matrix, which is ill-defined for CFTs. Therefore there is a logical possibility that the Coleman-Mandula theorem, instead of the  $a$ -theorem, is violated at the AD points under consideration.

In this respect, we think it worthwhile to stress that in a string/M-theory setup or in dimensional deconstruction it is quite common to have a new spacetime direction generated at a particular point in the moduli space, thus 'violating' the Coleman-Mandula theorem because of the appearance of a new symmetry which does not commute with the original spacetime symmetry. The failure of the theorem occurs exactly as anticipated by Coleman-Mandula [27] — by the appearance of an infinite number of light states, which are the Kaluza-Klein towers from the point of view of the lower-dimensional theory.

can mix, as is required for such proofs to work.

We hope that these remarks remove any doubts that we have indeed found a counterexample to the  $a$ -‘theorem’. Our finding highlights the peculiar dynamics of the AD points found in [10] when the ratio  $N_f/N_c$  is large. We would deem further study of these SCFTs and the flows between them worthwhile. The counterexample of lowest rank is the flow from the AD point of the  $SU(5)$  theory with six quarks, to the  $SU(4)$  theory with eight massless quarks close to the infinite coupling point. Now, the  $SU(3)$  theory with six massless quarks is known to be dual to an  $SU(2)$  gauge theory coupled to an  $N_f = 1$  fundamental hypermultiplet and to the exceptional rank-1 SCFT with flavor symmetry  $E_6$  [28]. Extending this duality to the  $SU(4)$  theory with eight quarks might shed new light on the dynamics of the flow between these two superconformal points.

## Acknowledgments

The authors would like to thank P. Argyres, V. Balasubramanian, M. R. Douglas, T. Eguchi, K. Intriligator, D. Martelli, Y. Nakayama, N. Seiberg, B. Wecht, and E. Witten for illuminating discussions. They would like to thank P. Argyres in particular for careful reading of the manuscript and for his valuable comments. They would also like to thank I. V. Melnikov for pointing out errors in (1.5) and (4.1) in the initial e-print version.

AS gratefully acknowledges support from the Ambrose Monell Foundation and the Institute for Advanced Study. The work of AS is also partially supported by NSF grants PHY-0555444 and PHY-0245214. The work of YT is in part supported by the Carl and Toby Feinberg fellowship at the Institute for Advanced Study, and by the United States DOE Grant DE-FG02-90ER40542.

## A. Brief history of the $a$ -theorem

The question of whether a version of the  $c$ -theorem exists in four dimensions was raised by Cardy [29], who pointed out that the simplest generalization of Zamolodchikov’s  $c$ -function — constructed from the two-point function of the stress tensor — need not be monotonically decreasing along flows in more than two dimensions. Noting that the 2D  $c$ -function can be alternatively defined by

$$c \equiv -\frac{3}{\pi} \int_{S^2} \langle T_\mu^\mu \rangle \sqrt{g} d^2x \quad (\text{A.1})$$

he proposed defining a  $d$ -dimensional ‘ $c$ -function’ proportional to

$$\int_{S^d} \langle T_\mu^\mu \rangle \sqrt{g} d^d x \quad (\text{A.2})$$

In four dimensions, this definition reproduces the function  $a$  in (1.2), since the Weyl curvature of the 4-sphere vanishes. Furthermore, it naturally provides a definition of  $a$ -function away from the conformal point.

The  $c$ -theorem for perturbative fixed points was then proved e.g. in [30]. But the definitive modern approach to  $a$ -theorem for SCFTs<sup>2</sup> was initiated by Anselmi and his

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<sup>2</sup>There have been some analyses without the help of supersymmetry, see e.g. [31, 32].



collaborators, culminating in the papers [33, 2]. There it was shown how various central charges are related to coefficients of operator product expansions of the energy momentum tensors and R-currents. Also uncovered were the relations between 't Hooft anomalies of R-currents and the central charges.

Following these works, Intriligator and Wecht [3] discovered the  $a$ -maximization procedure, which fixes the  $U(1)_R$  symmetry as a linear combination of possible  $U(1)$  symmetries. Kutasov *et al.* [4] then showed how operators which apparently hit the unitarity bound can be dealt with by postulating the appearance of extra accidental  $U(1)$  symmetries, leading to a more general proof of the  $a$ -theorem [5, 6]. An implicit assumption of their approach is that there should be additional  $U(1)$  symmetries, defined along the entire flow, with which  $U(1)_R$  can mix.

The AdS/CFT correspondence [34] offers another approach to the  $a$ -theorem [14, 35]. At leading order in the  $1/N$  expansion, the central charges  $a$  and  $c$  are equal, and are related to the cosmological constant in the dual  $AdS_5$  space [13]. The RG flow is related to the flow of the scalars in the 5d space, which changes the 5d vacuum energy [36]. In [14] it was shown that the monotonic decrease in  $a$  follows from a suitable energy condition in the gravity dual.

One particularly interesting holographic manifestation of the  $a$ -theorem is the following [37] (see also section 2.2.3 of [38]). A natural class of six-dimensional Calabi-Yau cones is the set of generalized conifolds

$$C_n : \quad x^2 + y^2 + z^2 + w^n = 0. \quad (A.3)$$

The central charge  $a_n$  of the theory on  $N$  D3-branes placed at the origin of the cone  $C_n$  can be found by the methods of [39]; it satisfies  $a_n > a_{n+1}$  when  $n$  is sufficiently large. By considering a deformation of the cone  $C_{n+1}$  by  $\epsilon w^n$  and recalling that the radial direction corresponds to the energy scale, one concludes that the UV theory is  $C_{n+1}$  and the IR is  $C_n$ . This example thus seems to violate the  $a$ -theorem. However, although the generalized conifold  $C_n$  with  $n \geq 3$  is Calabi-Yau in the sense that nowhere-vanishing holomorphic 3-form exists, it does *not* admit a Ricci-flat metric, because it violates the so-called Bishop bound [37]. Thus, a direct contradiction with the  $a$ -theorem is avoided in this case, because this RG flow does not correspond to a valid supergravity solution.

Finally let us stress that our counterexample to the  $a$ -theorem does not mean the end of the quest for the right  $c$ -function for 4d CFTs, which measures the number of degrees of freedom. Indeed, there is still a good chance that some other quantity, like the ratio  $f(T)/T^4$  of the free energy density to the temperature to the fourth [40], might satisfy at least the weak form of the ' $c$ -theorem.'

## References

- [1] A.B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a 2D field theory*, *JETP Lett.* **43** (1986) 730 [*Pisma Zh. Eksp. Teor. Fiz.* **43** (1986) 565].
- [2] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, *Phys. Rev.* **D 57** (1998) 7570 [[hep-th/9711035](#)].

- [3] K. Intriligator and B. Wecht, *The exact superconformal R-symmetry maximizes a*, *Nucl. Phys. B* **667** (2003) 183 [[hep-th/0304128](#)].
- [4] D. Kutasov, A. Parnachev and D.A. Sahakyan, *Central charges and  $U(1)_R$  symmetries in  $N = 1$  super Yang-Mills*, *JHEP* **11** (2003) 013 [[hep-th/0308071](#)].
- [5] D. Kutasov, *New results on the 'a-theorem' in four dimensional supersymmetric field theory*, [hep-th/0312098](#).
- [6] E. Barnes, K.A. Intriligator, B. Wecht and J. Wright, *Evidence for the strongest version of the 4D a-theorem, via a-maximization along RG flows*, *Nucl. Phys. B* **702** (2004) 131 [[hep-th/0408156](#)].
- [7] A.D. Shapere and Y. Tachikawa, *Central charges of  $N = 2$  superconformal field theories in four dimensions*, *JHEP* **09** (2008) 109 [[arXiv:0804.1957](#)].
- [8] P.C. Argyres and M.R. Douglas, *New phenomena in  $SU(3)$  supersymmetric gauge theory*, *Nucl. Phys. B* **448** (1995) 93 [[hep-th/9505062](#)].
- [9] P.C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, *New  $N = 2$  superconformal field theories in four dimensions*, *Nucl. Phys. B* **461** (1996) 71 [[hep-th/9511154](#)].
- [10] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of  $N = 2$  superconformal field theories in 4 dimensions*, *Nucl. Phys. B* **471** (1996) 430 [[hep-th/9603002](#)].
- [11] T. Banks and A. Zaks, *On the phase structure of vector-like gauge theories with massless fermions*, *Nucl. Phys. B* **196** (1982) 189.
- [12] P.C. Argyres and J.R. Wittig, *Infinite coupling duals of  $N = 2$  gauge theories and new rank 1 superconformal field theories*, *JHEP* **01** (2008) 074 [[arXiv:0712.2028](#)].
- [13] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)].
- [14] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [[hep-th/9904017](#)].
- [15] S.M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in  $N = 2$  superconformal theory*, *Class. and Quant. Grav.* **17** (2000) 665 [[hep-th/9907107](#)].
- [16] E. Witten, *On S duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [[hep-th/9505186](#)].
- [17] G.W. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, *Adv. Theor. Math. Phys.* **1** (1998) 298 [[hep-th/9709193](#)].
- [18] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys. B* **534** (1998) 549 [[hep-th/9711108](#)].
- [19] M. Mariño and G.W. Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, *Commun. Math. Phys.* **199** (1998) 25 [[hep-th/9802185](#)].
- [20] M. Mariño, G.W. Moore and G. Peradze, *Superconformal invariance and the geography of four-manifolds*, *Commun. Math. Phys.* **205** (1999) 691 [[hep-th/9812055](#)].
- [21] P.C. Argyres, M.R. Plesser and A.D. Shapere, *The Coulomb phase of  $N = 2$  supersymmetric QCD*, *Phys. Rev. Lett.* **75** (1995) 1699 [[hep-th/9505100](#)].

- [22] A. Hanany and Y. Oz, *On the quantum moduli space of vacua of  $N = 2$  supersymmetric  $SU(N_c)$  gauge theories*, *Nucl. Phys. B* **452** (1995) 283 [[hep-th/9505075](#)].
- [23] P.C. Argyres, M.R. Plesser and N. Seiberg, *The moduli space of  $N = 2$  SUSY QCD and duality in  $N = 1$  SUSY QCD*, *Nucl. Phys. B* **471** (1996) 159 [[hep-th/9603042](#)].
- [24] D.M. Hofman and J. Maldacena, *Conformal collider physics: energy and charge correlations*, *JHEP* **05** (2008) 012 [[arXiv:0803.1467](#)].
- [25] O. Aharony and Y. Tachikawa, *A holographic computation of the central charges of  $D = 4$ ,  $N = 2$  SCFTs*, *JHEP* **01** (2008) 037 [[arXiv:0711.4532](#)].
- [26] S. Weinberg, *The quantum theory of fields*, vol. III, Cambridge U.K. (2000).
- [27] S.R. Coleman and J. Mandula, *All possible symmetries of the  $S$  matrix*, *Phys. Rev.* **159** (1967) 1251.
- [28] P.C. Argyres and N. Seiberg,  *$S$ -duality in  $N = 2$  supersymmetric gauge theories*, *JHEP* **12** (2007) 088 [[arXiv:0711.0054](#)].
- [29] J.L. Cardy, *Is there a  $c$ -theorem in four-dimensions?*, *Phys. Lett. B* **215** (1988) 749.
- [30] I. Jack and H. Osborn, *Analogs for the  $c$ -theorem for four-dimensional renormalizable field theories*, *Nucl. Phys. B* **343** (1990) 647.
- [31] S. Forte and J.I. Latorre, *A proof of the irreversibility of renormalization group flows in four dimensions*, *Nucl. Phys. B* **535** (1998) 709 [[hep-th/9805015](#)].
- [32] A. Cappelli, R. Guida and N. Magnoli, *Exact consequences of the trace anomaly in four dimensions*, *Nucl. Phys. B* **618** (2001) 371 [[hep-th/0103237](#)].
- [33] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys. B* **526** (1998) 543 [[hep-th/9708042](#)].
- [34] J.M. Maldacena, *The large- $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [35] E. Álvarez and C. Gómez, *Geometric holography, the renormalization group and the  $c$ -theorem*, *Nucl. Phys. B* **541** (1999) 441 [[hep-th/9807226](#)].
- [36] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of  $N = 4$  super Yang-Mills from  $AdS$  dynamics*, *JHEP* **12** (1998) 022 [[hep-th/9810126](#)].
- [37] J.P. Gauntlett, D. Martelli, J. Sparks and S.-T. Yau, *Obstructions to the existence of Sasaki-Einstein metrics*, *Commun. Math. Phys.* **273** (2007) 803 [[hep-th/0607080](#)].
- [38] Y. Nakayama, *Black hole - string transition and rolling  $D$ -brane*, [hep-th/0702221](#).
- [39] A. Bergman and C.P. Herzog, *The volume of some non-spherical horizons and the  $AdS/CFT$  correspondence*, *JHEP* **01** (2002) 030 [[hep-th/0108020](#)].
- [40] T. Appelquist, A.G. Cohen and M. Schmaltz, *A new constraint on strongly coupled field theories*, *Phys. Rev. D* **60** (1999) 045003 [[hep-th/9901109](#)].